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An application of a homotopy analysis method to nonlinear composites

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Abstract

In this paper, the homotopy analysis method is employed to decompose the classical boundary-value problem of nonlinear composite media into an infinite number of linear partial differential equations, solving each linear partial differential equation by separating variables together with the method of undetermined coefficients. Although the calculation is rather complicated, the solutions obtained in this paper have much higher precision than those known ones. We also investigate the effective conductivity of nonlinear composite media, and obtain an approximate expression for it.

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1. Introduction

There has been growing interest in the physics of nonlinear inhomogeneous media because of their potential applications in engineering and science [1–8]. In particular, much effort has been centered around the calculations of the effective response in nonlinear composite systems consisting of two or more materials of different dielectric functions or conductivities [1, 7–15]. According to whether or not the linear part is dominant in the constitutive relation, two important limits are studied: weakly nonlinear and strongly nonlinear composites. The perturbation method is a powerful tool to deal with the weakly nonlinear case for the reason that weak nonlinearity can be treated as a small perturbation. However, it is unlucky that it does not work for the strongly nonlinear case. The homotopy analysis method (HAM) [16–20] is effective in dealing with strongly nonlinear equations. Recently, Liu and Li [21]

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considered a model nonlinear composite with a cylindrical inclusion embedded in a host medium. They employed the HAM to study such a system with the more general J - E relation of the form $J = \sigma_\alpha E + \chi_\alpha |E|^2 E$, where σ and χ are conductive coefficients, $\alpha = i$ or m indicates inclusions or host medium, respectively. For such a mixed nonlinear system, the static Maxwell equations $\nabla \times E = 0$ and $\nabla \cdot J = 0$ lead to the following equation:

$$\nabla \cdot [-\sigma_\alpha \nabla \phi - \chi_\alpha |\nabla \phi|^2 \nabla \phi] = 0, \quad (1)$$

where ϕ is the potential which satisfies $E = -\nabla \phi$. The boundary conditions for the continuity of the potential and the current density must be applied on the surfaces of inclusions

$$\phi^m = \phi^i \quad \text{on} \quad \partial\Omega_i, \quad (2)$$

$$\hat{n} \cdot J^m = \hat{n} \cdot J^i \quad \text{on} \quad \partial\Omega_i \quad (\text{from } \nabla \cdot J = 0), \quad (3)$$

where $\partial\Omega_i$ denotes the surface of the inclusion.

It is worth mentioning that in [21], for simplicity of calculation, the mode expansion method was first used to separate variables, namely, by assuming the electric potential in the form

$$\phi^\alpha(r, \theta) = \phi_1^\alpha(r) \cos \theta + \phi_3^\alpha(r) \cos 3\theta + \dots, \quad \alpha = i, m. \quad (4)$$

Substituting (4) into the governing equation (1) as well as into the boundary conditions (2) and (3) gives the resulting differential equations and the boundary conditions as follows:

$$\begin{aligned} \sum_n^\infty h_n^\alpha(r) \cos(2n-1)\theta &= 0, \quad \text{in } \Omega_\alpha, \quad \alpha = i, m, \\ \sum_n^\infty \chi_i g_n^i(r) \cos(2n-1)\theta &= \sum_n^\infty \chi_m g_n^m(r) \cos(2n-1)\theta|_{r=\rho}, \\ \sum_n^\infty \phi_n^i(r) \cos(2n-1)\theta &= \sum_n^\infty \phi_n^m(r) \cos(2n-1)\theta|_{r=\rho}, \end{aligned}$$

where $h_n^\alpha(r)$ and $g_n^\alpha(r)$ are the functions of $\phi_n^\alpha(r)$ and their derivatives. Thus an ordinary system is obtained

$$h_n^i(r) = 0, \quad \text{in } \Omega_i, \quad n = 1, 3, \dots, \quad (5)$$

$$h_n^m(r) = 0, \quad \text{in } \Omega_m, \quad n = 1, 3, \dots, \quad (6)$$

$$\chi_i g_n^i(r) = \chi_m g_n^m(r)|_{r=\rho}, \quad n = 1, 3, \dots, \quad (7)$$

$$\phi_n^i(r) = \phi_n^m(r)|_{r=\rho}, \quad n = 1, 3, \dots \quad (8)$$

Furthermore, they truncate the system by retaining only the first mode to keep the lowest approximation. The resulting equations and the boundary conditions read

$$\sigma_i F_1[\phi^i] + \chi_i F_3[\phi^i] = 0, \quad (9)$$

$$\sigma_m F_1[\phi^m] + \chi_m F_3[\phi^m] = 0, \quad (10)$$

$$\phi^i(r) = \phi^m(r)|_{r=\rho}, \quad (11)$$

$$\sigma_i J_1[\phi^i] + \chi_i J_3[\phi^i] = \sigma_m J_1[\phi^m] + \chi_m J_3[\phi^m]|_{r=\rho}, \quad (12)$$

where

$$\begin{aligned}
 F_1[\phi] &= (4r^3\phi_r + 4r^4\phi_{rr} - 4r^2\phi)/(4r^4), \\
 F_3[\phi] &= (r^2\phi^2\phi_{rr} - 3\phi^3 + r^2\phi\phi_r^2 - r\phi^2\phi_r + 9r^4\phi_r^2\phi_{rr} + 3r^3\phi_r^3)/(4r^4), \\
 J_1[\phi] &= \phi_r, \quad J_3[\phi] = (1/4r^2)\phi^2\phi_r + 3\phi_r^3/4.
 \end{aligned}$$

As there will be no confusion in subsequent discussions, the subscript 1 in (9)–(12) is omitted.

Although the obtained solutions in [21] have much higher precision for the system (9)–(12), they still keep the lowest approximation w.r.t. r and θ for the original system (1)–(3). In this paper, we solve the system (1)–(3) again by directly using the HAM. In this case, the obtained higher order deformation equations are linear partial differential ones. Solving each of them by separating variables together with the method of undetermined coefficients gives much more analytical approximate solutions.

The paper is organized as follows. In section 2, we briefly introduce the basic idea of the HAM. In section 3, the HAM is employed to solve the original system (1)–(3). In section 4, the convergence of the homotopy analysis solutions is analyzed. In section 5, the effective conductivity is calculated and formulated. Finally, a brief summary is given.

2. The homotopy analysis method

Based on homotopy, which is a basic concept in topology, a new analytic method (namely, the HAM) is proposed to obtain series solutions of nonlinear differential equations. Different from perturbation techniques, this approach is independent of any small/large physical parameters. So, it is valid not only for weakly nonlinear systems but also for strongly nonlinear cases. Furthermore, as shown in [20], the HAM logically contains Lyapunov’s small parameter method, the δ -expansion method and Adomian’s decomposition method, and therefore unifies these nonperturbation methods and is more general than them. Besides, different from all previous analytic methods, the HAM provides us with a simple way to adjust and control the convergence of solution series. Especially, it provides us with great freedom to replace a nonlinear differential equation of order n into an infinite number of linear differential equations of order k , where the order k is even unnecessary to be equal to the order n .

The basic idea of the HAM is as follows.

2.1. Rule of solution expression

When the HAM is applied to solve a nonlinear differential system, such as

$$N[u(r, t)] = 0, \tag{13}$$

where N is a nonlinear operator.

First, a set of base functions

$$\{e_n(r, t) \mid n = 0, 1, 2, 3, \dots\}$$

are selected to represent the required solution

$$u(r, t) = \sum_{n=0}^{\infty} c_n e_n(r, t),$$

where c_n is a coefficient. This is the so-called rule of solution expression. We should emphasize two facts. First, a solution of a nonlinear problem may be expressed by different sets of base functions. Second, in many cases, from physical characteristics and boundary/initial conditions, it is often not very difficult to determine the type of base functions convenient

to represent solutions of a given nonlinear problem. But how can we know that a set of base functions is better than others and is more efficient to approximate a nonlinear problem? Unfortunately, up to now, the rule of solution expression implies such an assumption that we should have, more or less, some knowledge about a given nonlinear problem *a priori*. Fortunately, the so-called homotopy-Padé technique can greatly enlarge the convergence region and rate of solution series.

2.2. *Choosing initial guess and auxiliary linear operator*

It is under the rule of solution expression that the initial approximation and the auxiliary linear operator are selected. The initial guess $u_0(r, t)$ must be expressed by a sum of the base functions, i.e.

$$u_0(r, t) = \sum_{n=0}^{M_0} a_n e_n(r, t), \tag{14}$$

where a_n is a coefficient and M_0 is an integer.

Similarly, the auxiliary linear operator L must be chosen in such a way that the solution to the equation

$$L[w(r, t)] = 0 \tag{15}$$

must be expressed by a sum of the base functions, say,

$$w(r, t) = \sum_{n=0}^{M_1} b_n e_n(r, t), \tag{16}$$

where b_n is a coefficient and M_1 is an integer. Moreover, L has the property $L[w(r, t)] = 0$ when $w(r, t) = 0$.

2.3. *The zero-order deformation equation*

Using $q \in [0, 1]$ as an embedding parameter, we construct a new kind of homotopy with the form

$$\mathcal{H}(\Phi(r, t, q); q, h, H(r, t)) = (1 - q)\{L[\Phi(r, t, q) - u_0(r, t)]\} - qhH(r, t)N[\Phi(r, t, q)], \tag{17}$$

in which $h \neq 0$ is an auxiliary parameter, $H(r, t) \neq 0$ is an auxiliary function. They play an important role in the homotopy analysis method.

Forcing the homotopy (17) to be zero, we obtain the so-called zero-order deformation equation

$$(1 - q)\{L[\Phi(r, t, q) - u_0(r, t)]\} = qhH(r, t)N[\Phi(r, t, q)]. \tag{18}$$

When $q = 0$, the zero-order deformation equation (18) becomes

$$L[\Phi(r, t, 0) - u_0(r, t)] = 0, \tag{19}$$

which gives

$$\Phi(r, t, 0) = u_0(r, t). \tag{20}$$

When $q = 1$, since $h \neq 0$ and $H(r, t) \neq 0$, the zero-order deformation equation (18) is equivalent to

$$N[\Phi(r, t, 1)] = 0, \tag{21}$$

which is exactly the same as the original equation (13), provided

$$\Phi(r, t, 1) = u(r, t). \tag{22}$$

Thus, according to (20) and (22), as the embedding parameter q increases from 0 to 1, $\Phi(r, t, q)$ varies continuously from the initial guess $u_0(r, t)$ to the exact solution $u(r, t)$ of the original equation (13).

Due to Taylor's theorem and (20), we expand $\Phi(r, t, q)$ in the power series

$$\Phi(r, t, q) = u_0(r, t) + \sum_{n=1}^{\infty} u_n(r, t)q^n, \tag{23}$$

where

$$u_n(r, t) = \frac{1}{n!} \left. \frac{\partial^n \Phi(r, t, q)}{\partial q^n} \right|_{q=0}. \tag{24}$$

Assuming that the above series is convergent when $q = 1$, we have due to (20) that

$$u(r, t) = u_0(r, t) + \sum_{n=1}^{\infty} u_n(r, t). \tag{25}$$

2.4. The higher order deformation equation

Differentiating the zero-order deformation equation (18) n times with respect to q and then dividing them by $n!$ and finally setting $q = 0$, we have the so-called n th-order deformation equation

$$L[u_n(r, t) - \chi_n u_{n-1}(r, t)] = hH(r, t)R_n(\vec{u}_{n-1}, r, t), \tag{26}$$

where

$$R_n(\vec{u}_{n-1}, r, t) = \frac{1}{(n-1)!} \left. \frac{\partial^{n-1} N[\Phi(r, t, q)]}{\partial q^{n-1}} \right|_{q=0} \tag{27}$$

and

$$\chi_n = \begin{cases} 0, & n \leq 1, \\ 1, & n > 1. \end{cases} \tag{28}$$

The n th-order deformation equation (26) is linear. By properly choosing initial guess $u_0(r, t)$, the auxiliary linear operator L and the auxiliary function $H(r, t)$, equation (26) can easily be solved, especially by means of symbolic computation software such as Mathematica, Maple and so on.

2.5. The convergence theorem

As proved by Liao [20] in general, if h is properly chosen (as shown in figure 1, the value of h is determined by means of the so-called h curve) so that the series (23) is convergent at $q = 1$, one can get as accurate approximations as possible by means of the series (25), i.e. we have the following.

Theorem (convergence theorem). *As long as the series*

$$u_0(r, t) + \sum_{n=1}^{\infty} u_n(r, t)$$

is convergent, where $u_n(r, t)$ is governed by the higher order deformation equation (26) under definitions (27) and (28), it must be a solution of equation (13).

3. Homotopy analysis solution

In this section, we apply the HAM to solve the system (1)–(3).

To meet the asymptotic property

$$\phi^i(0, 0) = 0, \quad \phi^m(\infty, 0) = -E_0r, \quad (29)$$

we can express ϕ^m in the form

$$\phi^m(r, \theta) = ar \cos \theta + \sum_{j=1}^{\infty} f_j(r) \cos(j\theta), \quad (30)$$

and ϕ^i in the form

$$\phi^i(r, \theta) = \sum_{j=1}^{\infty} g_j(r) \cos(j\theta), \quad (31)$$

where a is a coefficient, $f_j(r)$ is a rational proper fraction w.r.t. r , $g_j(r)$ is a polynomial w.r.t. r . This provides us with the so-called rule of solution expression.

Under the rule of solution expression denoted by (30) and (31) as well as the property (29), it is straightforward to choose the initial guess

$$\phi^m_0(r, \theta) = -\frac{E_0d_0 \cos \theta}{r} \quad (32)$$

and

$$\phi^i_0(r, \theta) = -E_0c_0r \cos \theta, \quad (33)$$

where d_0 and c_0 are parameters to be determined later. E_0 is an external electric field. According to the rule of solution expression and from (1), we choose the auxiliary linear operators as

$$L[\phi^\alpha(r, \theta)] = -\frac{1}{r^2}(r\phi^\alpha_r + \phi^\alpha_{\theta\theta} + r^2\phi^\alpha_{rr}), \quad \alpha = i, m, \quad (34)$$

with the property $L[\phi^\alpha(r, \theta)] = 0$ when $\phi^\alpha(r, \theta) = 0$.

It is to be stressed that the operator (34) is composed by the linear part of equation (1). However, in the HAM the linear operator L may have nothing to do with the original system.

Following the method of separating variables, for equation (34), by assuming $\phi^\alpha(r, \theta) = f(r)g(\theta)$, we have

$$f_{rr} + \frac{1}{r}f_r - \frac{c^2}{r^2}f = 0, \quad g_{\theta\theta} + c^2g = 0. \quad (35)$$

The solutions to (35) can easily be obtained as

$$f = C_1r^c + C_2r^{-c}, \quad g = C_3 \cos c\theta + C_4 \sin c\theta. \quad (36)$$

Letting $C_4 = 0$, the obtained solution $\phi^m(r, \theta) = f(r)g(\theta) = (\tilde{C}_1r^c + \tilde{C}_2r^{-c}) \cos c\theta$ coincides with (30). Similarly, letting $C_2 = C_4 = 0$, the obtained solution $\phi^i(r, \theta) = f(r)g(\theta) = \tilde{C}_1r^c \cos c\theta$ coincides with (31).

By mapping $u(r, t) \rightarrow \Phi(r, t, q)$, the nonlinear operator (1) is rewritten as

$$\begin{aligned} N[\Phi^\alpha(r, \theta; q)] = & -[\sigma_\alpha r^3 \Phi^\alpha_r + \chi_\alpha r^3 (\Phi^\alpha_r)^3 - r \chi_\alpha \Phi^\alpha_r (\Phi^\alpha_\theta)^2 + r^4 \sigma_\alpha \Phi^\alpha_{rr} \\ & + 3r^4 \chi_\alpha \Phi^\alpha_{rr} (\Phi^\alpha_r)^2 + r^2 \chi_\alpha \Phi^\alpha_{rr} (\Phi^\alpha_\theta)^2 + 4\chi_\alpha r^2 \Phi^\alpha_\theta \Phi^\alpha_r \Phi^\alpha_{r\theta} \\ & + \sigma_\alpha r^2 \Phi^\alpha_{\theta\theta} + \chi_\alpha r^2 \Phi^\alpha_{\theta\theta} (\Phi^\alpha_r)^2 + 3\chi_\alpha \Phi^\alpha_{\theta\theta} (\Phi^\alpha_\theta)^2] / r^4, \quad \alpha = i, m, \end{aligned} \quad (37)$$

where $q \in [0, 1]$ is an embedding parameter. Let $h_1 (\neq 0)$, $h_2 (\neq 0)$ denote nonzero auxiliary parameters and $H_1(r, \theta)$, $H_2(r, \theta)$ nonzero auxiliary functions, respectively. The zero-order deformation equations in the host and in the inclusion constructed by us respectively are

$$(1 - q)L[\Phi^m(r, \theta, q) - \phi^m_0(r, \theta)] = qh_1H_1(r, \theta)N[\Phi^m(r, \theta, q)] \quad (38)$$

and

$$(1 - q)L[\Phi^i(r, \theta, q) - \phi^i_0(r, \theta)] = qh_2H_2(r, \theta)N[\Phi^i(r, \theta, q)], \quad (39)$$

which are subject to the boundary conditions

$$\Phi^m(0, 0, q) = 0, \quad \left. \frac{\partial \Phi^m(r, 0, q)}{\partial r} \right|_{r=+\infty} = -E_0. \quad (40)$$

When $q = 0$, it is easy to verify that

$$\Phi^m(r, \theta, 0) = \phi^m_0(r, \theta), \quad \Phi^i(r, \theta, 0) = \phi^i_0(r, \theta). \quad (41)$$

When $q = 1$, since $h_1 \neq 0$, $h_2 \neq 0$, $H_1(r, \theta) \neq 0$ and $H_2(r, \theta) \neq 0$, the zero-order deformation equations (38)–(40) are equivalent to the original equations (1)–(3), provided

$$\Phi^m(r, \theta, 1) = \phi^m(r, \theta), \quad \Phi^i(r, \theta, 1) = \phi^i(r, \theta). \quad (42)$$

Thus, as q increases from 0 to 1, $\Phi^m(r, \theta, q)$ and $\Phi^i(r, \theta, q)$ vary from the initial guesses $\phi^m_0(r, \theta)$, $\phi^i_0(r, \theta)$ to the solutions $\phi^m(r, \theta)$, $\phi^i(r, \theta)$ of equations (1)–(3), respectively.

By Taylor’s theorem and using (41), we obtain the power series

$$\Phi^m(r, \theta, q) = \phi^m_0(r, \theta) + \sum_{n=1}^{\infty} \phi^m_n(r, \theta)q^n \quad (43)$$

and

$$\Phi^i(r, \theta, q) = \phi^i_0(r, \theta) + \sum_{n=1}^{\infty} \phi^i_n(r, \theta)q^n, \quad (44)$$

where

$$\phi^m_n(r, \theta) = \frac{1}{n!} \left. \frac{\partial^n \Phi^m(r, \theta, q)}{\partial q^n} \right|_{q=0}, \quad \phi^i_n(r, \theta) = \frac{1}{n!} \left. \frac{\partial^n \Phi^i(r, \theta, q)}{\partial q^n} \right|_{q=0}. \quad (45)$$

Assuming that the auxiliary parameters h_1 , h_2 and the auxiliary functions $H_1(r, \theta)$, $H_2(r, \theta)$ are properly chosen so that the above series converge at $q = 1$, we have

$$\phi^m(r, \theta) = \phi^m_0(r, \theta) + \sum_{n=1}^{\infty} \phi^m_n(r, \theta) \quad (46)$$

and

$$\phi^i(r, \theta) = \phi^i_0(r, \theta) + \sum_{n=1}^{\infty} \phi^i_n(r, \theta). \quad (47)$$

The corresponding M th-order approximations are given by

$$\phi^m(r, \theta) \approx \phi^m_0(r, \theta) + \sum_{n=1}^M \phi^m_n(r, \theta), \quad (48)$$

$$\phi^i(r, \theta) \approx \phi^i_0(r, \theta) + \sum_{n=1}^M \phi^i_n(r, \theta). \quad (49)$$

Note that $\phi_n^m(r, \theta)$ and $\phi_n^i(r, \theta)$, ($n = 0, 1, 2, \dots$) satisfy

$$\phi_n^i(r, \theta) = \phi_n^m(r, \theta)|_{r=\rho} \tag{50}$$

and

$$\frac{\partial^n J[\Phi^m(r, \theta, q)]}{\partial q^n} = \frac{\partial^n J[\Phi^i(r, \theta, q)]}{\partial q^n} \Big|_{q=0, r=\rho}, \tag{51}$$

where $J[\Phi^\alpha(r, \theta, q)] = \Phi_r^\alpha [\sigma_\alpha r^2 + \chi_\alpha (r^2 (\Phi_r^\alpha)^2 + (\Phi_\theta^\alpha)^2)] / r^2$, $\alpha = i$ or m . From (43) and (44), it can be seen that the boundary conditions (2) and (3) are satisfied.

It is to be stressed that the asymptotic conditions (29) are a part of the boundary conditions for the problem we discussed. As our obtained solutions can be expressed in the form of (30) and (31), respectively, we can see that the asymptotic conditions (29) are satisfied. So in the following computation and analysis, we just consider the boundary conditions (2) and (3), conditions (50) and (51) are just components form of (2) and (3), respectively.

For the sake of simplicity, we define the vectors

$$\vec{\phi}_k^m(r, \theta) = (\phi_0^m(r, \theta), \phi_1^m(r, \theta), \phi_2^m(r, \theta), \dots, \phi_k^m(r, \theta))$$

and

$$\vec{\phi}_k^i(r, \theta) = (\phi_0^i(r, \theta), \phi_1^i(r, \theta), \phi_2^i(r, \theta), \dots, \phi_k^i(r, \theta)).$$

Differentiating the zero-order deformation equations (38) and (39) k times w.r.t. q , then dividing by $k!$, and finally setting $q = 0$, we have the higher order deformation equations

$$L[\phi_k^m(r, \theta) - \chi_k \phi_{k-1}^m(r, \theta)] = h_1 H_1(r, \theta) R_k^m(\vec{\phi}_{k-1}^m(r, \theta)), \tag{52}$$

$$L[\phi_k^i(r, \theta) - \chi_k \phi_{k-1}^i(r, \theta)] = h_2 H_2(r, \theta) R_k^i(\vec{\phi}_{k-1}^i(r, \theta)), \tag{53}$$

which are subject to the boundary conditions

$$\phi_k^i(0, 0) = 0, \quad \frac{\partial \phi_k^m(r, 0)}{\partial r} \Big|_{r=+\infty} = 0, \tag{54}$$

where

$$R_k^m(\vec{\phi}_{k-1}^m(r, \theta)) = \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial q^{k-1}} N \left[\sum_{n=0}^{\infty} \phi_n^m(r, \theta) q^n \right] \Big|_{q=0},$$

$$R_k^i(\vec{\phi}_{k-1}^i(r, \theta)) = \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial q^{k-1}} N \left[\sum_{n=0}^{\infty} \phi_n^i(r, \theta) q^n \right] \Big|_{q=0}$$

and

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k > 1. \end{cases} \tag{55}$$

Note that the higher order deformation equations (52)–(54) are uncoupled, and each one is an inhomogeneous linear partial differential equation. With the aid of symbolic computation software, the solutions of them can be constructed by separating variables together with the method of undetermined coefficients. For simplicity, we set $H_1(r, \theta) = H_2(r, \theta) = 1$, $h_1 = h_2 = h$, and the critical value $\rho = 1$. Here an example is given to show how to solve each

higher order deformation equation; for example, the higher order deformation equation for $\phi^m_2(r, \theta)$ is

$$\begin{aligned} \sigma_m \left(\left(\frac{\partial^2}{\partial r^2} \phi^m_2(r, \theta) \right) r^2 + \left(\frac{\partial}{\partial r} \phi^m_2(r, \theta) \right) r + \left(\frac{\partial^2}{\partial \theta^2} \phi^m_2(r, \theta) \right) \right) \\ = \left(\frac{24E_0^5 d_0^2 d_1 h \chi_m}{r^7} - \frac{2h^2 E_0^5 d_0^5 \chi_m^2}{\sigma_m r^9} \right) \cos 3\theta - \left(\frac{8E_0^5 d_0^2 h \chi_m u_1}{r^3} \right. \\ \left. + \frac{12E_0^3 \sigma_m d_0^3 h \chi_m - 36E_0^5 \sigma_m d_0^2 b_1 h \chi_m + 12E_0^3 \sigma_m d_0^3 h^2 \chi_m}{3\sigma_m r^5} \right. \\ \left. - \frac{24E_0^5 d_0^2 d_1 h \chi_m}{r^7} + \frac{56h^2 E_0^5 d_0^5 \chi_m^2}{3\sigma_m r^9} \right) \cos \theta. \end{aligned} \tag{56}$$

Supposing that it has particular solutions

$$\phi^m_2(r, \theta) = \frac{s_1 \cos 3\theta}{r^7} + \frac{s_2 \cos 3\theta}{r^9 \sigma_m} + \frac{s_3 \cos \theta}{r^3} + \frac{s_4 \cos \theta}{r^5 \sigma_m} + \frac{s_5 \cos \theta}{r^7} + \frac{s_6 \cos \theta}{r^9 \sigma_m}, \tag{57}$$

and substituting (57) into (56), and then setting the coefficient of different items $\cos(i\theta)/r^j$ to be zero generates a linear algebraic system for s_k

$$\begin{cases} 4\sigma_m(3E_0^5 d_0^2 b_1 h \chi_m - 6s_4 - E_0^3 d_0^3 h^2 \chi_m - E_0^3 d_0^3 h \chi_m) = 0, \\ -8\sigma_m(-3h E_0^5 d_0^2 \chi_m d_1 + 5\sigma_m s_1) = 0, \\ -72\sigma_m s_2 - 2h^2 E_0^5 d_0^5 \chi_m^2 = 0, \\ 24\sigma_m(-2s_5 \sigma_m + h E_0^5 d_0^2 \chi_m d_1) = 0, \\ -56h^2 E_0^5 d_0^5 \chi_m^2 - 80\sigma_m s_6 = 0, \\ -8\sigma_m(s_3 \sigma_m + h E_0^5 d_0^2 \chi_m u_1) = 0, \end{cases} \tag{58}$$

which on solving gives

$$\begin{cases} s_1 = \frac{3h E_0^5 d_0^2 \chi_m d_1}{5\sigma_m}, s_2 = -\frac{h^2 E_0^5 d_0^5 \chi_m^2}{36\sigma_m}, s_3 = -\frac{h E_0^5 d_0^2 \chi_m u_1}{\sigma_m}, \\ s_4 = \frac{1}{2} E_0^5 d_0^2 b_1 h \chi_m - \frac{1}{6} E_0^3 d_0^3 h^2 \chi_m - \frac{1}{6} E_0^3 d_0^3 h \chi_m, \\ s_5 = \frac{h E_0^5 d_0^2 \chi_m d_1}{2\sigma_m}, s_6 = -\frac{7h^2 E_0^5 d_0^5 \chi_m^2}{10\sigma_m} \end{cases}. \tag{59}$$

Substituting (59) into (57), the particular solution of (56) reads

$$\begin{aligned} \phi^m_2(r, \theta) = E_0^3 d_0^2 h \chi_m (108 E_0^2 d_1 r^2 \sigma_m \cos(3\theta) - 5 h E_0^2 d_0^3 \chi_m \cos(3\theta) \\ - 180 E_0^2 \cos(\theta) u_1 r^6 \sigma_m + 90 \cos(\theta) r^4 \sigma_m E_0^2 b_1 \\ - 30 \cos(\theta) r^4 \sigma_m d_0 h - 30 \cos(\theta) r^4 \sigma_m d_0 + 90 E_0^2 d_1 r^2 \sigma_m \cos(\theta) \\ - 42 h E_0^2 d_0^3 \cos(\theta) \chi_m) / (180 r^9 \sigma_m^2). \end{aligned} \tag{60}$$

From the solution (36), we know that the corresponding homogeneous equation (34) has the solution $\phi^m(r, \theta) = br^c \cos c\theta + dr^{-c} \cos c\theta$ when $\alpha = m$. For each component $\phi^m_k(r, \theta)$ of the solution series, we determine the value of c according to the asymptotic condition (29) as well as the rules offered in the homotopy analysis method, namely the rule of solution expression (30) and the rule of coefficient ergodicity, which means that as the order of approximation tends to infinity, each case should appear in the solution expression and each coefficient can be modified. Therefore, we take $c = 1$ in the first component ($\phi^m_0(r, \theta)$)

only contains $\cos \theta$), in the second component ($\phi^m_1(r, \theta)$ contains $\cos \theta, \cos 3\theta$), taking $c = 1$ to build the $\cos \theta$ term, and $c = 3$ to build the $\cos 3\theta$ term, With the same idea, each component $\phi^i_k(r, \theta)$ can be uniquely determined.

On the other hand, to meet the asymptotic condition (29), the solution expression for ϕ^m in (30) contains a term $ar \cos(\theta)$. However, when we choose

$$\phi^m_0(r, \theta) = -\frac{E_0 d_0 \cos \theta}{r} + b_0 r \cos(\theta),$$

the obtained value of b_0 is zero in each valid solution for b_0, d_0, c_0 obtained by meeting the boundary conditions (50) and (51). Each of the conductive coefficients χ_m, χ_i, σ_m and σ_i should be positive. Therefore, we straightly choose $\phi^m_0(r, \theta) = -\frac{E_0 d_0 \cos \theta}{r}$, and assume that a term $u_1 r \cos(\theta)$ be contained in $\phi^m_1(r, \theta)$. From the following BC_1 we know that u_1 is arbitrary. For simplicity, we choose $u_1 = -E_0$; in such case, the asymptotic condition (29) is satisfied, and other components of solution series for ϕ^m do not need to contain a term which is linear w.r.t. $r \cos(\theta)$.

So the solution of the corresponding homogeneous equation of equation (56) is in the form

$$\phi^m_2(r, \theta) = E_0^5 \left(\frac{d_2}{r} \cos \theta + \frac{b_2}{r^3} \cos 3\theta + \frac{u_2}{r^5} \cos 5\theta \right), \quad (61)$$

where b_2, d_2, u_2 are parameters to be determined later.

The solution to equation (56) can be generated by combining solution (61) with (60).

In this way, we have

$$\begin{aligned} \phi^m_1(r, \theta) &= E_0^3 \left(\frac{b_1}{r} + \frac{h \chi_m d_0^3}{6r^5 \sigma_m} + u_1 r \right) \cos \theta + \frac{E_0^3 d_1}{r^3} \cos 3\theta, \\ \phi^i_1(r, \theta) &= E_0^3 c_1 r \cos \theta + E_0^3 e_1 r^3 \cos 3\theta, \\ \phi^m_2(r, \theta) &= \left(\frac{E_0^5 d_2}{r} - \frac{E_0^5 h \chi_m d_0^2 u_1}{\sigma_m r^3} + \frac{E_0^5 h \chi_m d_0^2 d_1}{2\sigma_m r^7} + \frac{7E_0^5 h^2 \chi_m^2 d_0^5}{30\sigma_m^2 r^9} \right. \\ &\quad \left. + \frac{E_0^3 h \chi_m d_0^2 (E_0^2 90b_1 \sigma_m + 30\sigma_m h d_0 + 30\sigma_m d_0)}{180\sigma_m^2 r^5} \right) \cos \theta \\ &\quad + E_0^5 \left(\frac{b_2}{r^3} + \frac{3h \chi_m d_0^2 d_1}{5\sigma_m r^7} + \frac{h^2 \chi_m^2 d_0^5}{36\sigma_m^2 r^9} \right) \cos 3\theta + \frac{E_0^5 u_2}{r^5} \cos 5\theta, \\ \phi^i_2(r, \theta) &= \left(E_0^5 c_2 r + \frac{3E_0^5 h \chi_i c_0^2 e_1}{2\sigma_i} r^3 \right) \cos \theta + E_0^5 e_2 r^3 \cos 3\theta + E_0^5 v_2 r^5 \cos 5\theta, \\ &\vdots \end{aligned}$$

To meet the boundary conditions (50) and (51) via the first component $\phi^m_0(r, \theta)$ and $\phi^i_0(r, \theta)$, we have

$$\begin{cases} \left(-\frac{E_0 d_0}{r} + E_0 c_0 r \right) \cos(\theta) = 0, \\ \frac{(-E_0 d_0 \sigma_m r^4 - E_0^3 d_0^3 \chi_m - E_0 c_0 \sigma_i r^6 - E_0^3 c_0^3 \chi_i r^6) \cos(\theta)}{r^6} = 0. \end{cases} \quad (62)$$

Letting the coefficient of $\cos(\theta)$ in each equation be zero, an algebraic equation for c_0, d_0 is obtained:

$$\begin{cases} -\frac{d_0}{r} + c_0 r = 0, \\ \frac{-d_0 \sigma_m r^4 - E_0^2 d_0^3 \chi_m - c_0 \sigma_i r^6 - E_0^2 c_0^3 \chi_i r^6}{r^6} = 0. \end{cases} \quad (63)$$

Solving it, we have $d_0 = c_0 = \pm \frac{\sqrt{(\chi_m - \chi_i)(\sigma_i - \sigma_m)}}{(\chi_m - \chi_i)E_0}$. Obviously, by exchanging χ_m and χ_i along with σ_m and σ_i , a solution with a negative sign can be generated. So we just take the positive sign and have

$$BC_0 = \left\{ c_0 = \frac{\sqrt{(\chi_i - \chi_m)(\sigma_m - \sigma_i)}}{(\chi_m - \chi_i)E_0}, d_0 = \frac{\sqrt{(\chi_i - \chi_m)(\sigma_m - \sigma_i)}}{(\chi_m - \chi_i)E_0} \right\}.$$

Similarly, by meeting the boundary condition (50) via the first and second components $\phi^m_0(r, \theta)$, $\phi^m_1(r, \theta)$ and $\phi^i_0(r, \theta)$, $\phi^i_1(r, \theta)$, we obtain

$$[6\sigma_m(c_1r^6 - b_1r^4 - u_1r^6) + d_0^3\chi_m h] \cos(\theta) + 6\sigma_m(e_1r^8 - d_1r^2) \cos(3\theta) = 0. \tag{64}$$

To meet the boundary condition (51), we have

$$\begin{aligned} & \left[3\chi_m r^2 \left(\frac{\partial}{\partial r} \phi^m_0(r, \theta) \right)^2 \left(\frac{\partial}{\partial r} \phi^m_1(r, \theta) \right) \right. \\ & + 2\chi_m \left(\frac{\partial}{\partial r} \phi^m_0(r, \theta) \right) \left(\frac{\partial}{\partial \theta} \phi^m_0(r, \theta) \right) \left(\frac{\partial}{\partial \theta} \phi^m_1(r, \theta) \right) \\ & \left. + \sigma_m r^2 \left(\frac{\partial}{\partial r} \phi^m_1(r, \theta) \right) + \chi_m \left(\frac{\partial}{\partial r} \phi^m_1(r, \theta) \right) \left(\frac{\partial}{\partial \theta} \phi^m_0(r, \theta) \right)^2 \right] / r^2 = 0. \end{aligned} \tag{65}$$

Substitute $\phi^m_0(r, \theta)$, $\phi^m_1(r, \theta)$, $\phi^i_0(r, \theta)$, $\phi^i_1(r, \theta)$ into (65), it has

$$\begin{aligned} & [-13E_0^2\chi_m^2d_0^5h + 18E_0^2\chi_md_0^2d_1r^2\sigma_m + 18E_0^2c_0^2\chi_ic_1r^{10}\sigma_m \\ & + 18E_0^2\chi_ic_0^2r^{12}e_1\sigma_m - 12E_0^2\chi_md_0^2u_1r^6\sigma_m + 18E_0^2d_0^2\chi_mb_1r^4\sigma_m \\ & - 5\sigma_mr^4d_0^3\chi_mh + 6\sigma_m^2r^8b_1 - 6\sigma_m^2r^{10}u_1 + 6\sigma_ic_1r^{10}\sigma_m] \cos(\theta) \\ & + [18d_1r^6\sigma_m^2 + 18\sigma_me_1r^{12}\sigma_i + 36E_0^2\chi_ic_0^2r^{12}e_1\sigma_m - 6E_0^2\chi_md_0^2u_1r^6\sigma_m \\ & + 36E_0^2\chi_md_0^2d_1r^2\sigma_m - 2E_0^2\chi_m^2d_0^5h] \cos(3\theta) = 0. \end{aligned} \tag{66}$$

Letting the coefficients of different terms $\cos(i\theta)$ in (64) and (66) be zero, an algebraic system for b_1, c_1, d_1, u_1, e_1 is obtained. It can be seen that equation (65) is linear for $\phi^m_1(r, \theta)$, $\phi^i_1(r, \theta)$ as well as their derivatives. Therefore the obtained system for b_1, c_1, d_1, u_1, e_1 is a group of linear algebraic equations. In this way, we have

$$\begin{aligned} BC_1 = \{ & e_1 = \chi_mE_0^2d_0^2(d_0^3\chi_mh + 3u_1\sigma_m) / [9\sigma_m(\sigma_m + 2E_0^2c_0^2\chi_i + 2\chi_mE_0^2d_0^2 + \sigma_i)], \\ & d_1 = \chi_mE_0^2d_0^2(d_0^3\chi_mh + 3u_1\sigma_m) / [9\sigma_m(\sigma_m + 2E_0^2c_0^2\chi_i + 2\chi_mE_0^2d_0^2 + \sigma_i)], \\ & c_1 = (9E_0^2\chi_md_0^2\sigma_m + 9E_0^4c_0^2\chi_i\chi_md_0^2 + 4E_0^2c_0^2\chi_i\sigma_m + 2\sigma_m\sigma_i \\ & + 9\chi_m^2E_0^4d_0^4 + 2\sigma_m^2 + 5\sigma_i\chi_mE_0^2d_0^2)(\sigma_m + 2E_0^2c_0^2\chi_i + 2\chi_mE_0^2d_0^2 \\ & + \sigma_i) / [3\sigma_m(\sigma_m + 2E_0^2c_0^2\chi_i + 2\chi_mE_0^2d_0^2 + \sigma_i)(\sigma_i + \sigma_m + 3\chi_mE_0^2d_0^2 \\ & + 3E_0^2c_0^2\chi_i)], \\ & b_1 = (24\chi_m^3E_0^4d_0^7h - 18E_0^4c_0^2\chi_i\chi_md_0^2u_1\sigma_m + 18\chi_m^2E_0^4d_0^4u_1\sigma_m \\ & + 23\sigma_m\chi_m^2E_0^2d_0^5h + 6\sigma_id_0^3\chi_m\sigma_mh + 6u_1\sigma_m^3 + 24E_0^2\chi_md_0^2u_1\sigma_m^2 \\ & - 6E_0^2c_0^2\chi_iu_1\sigma_m^2 + 30E_0^4c_0^2\chi_i\chi_m^2d_0^5h + 13E_0^2c_0^2\chi_id_0^3\chi_m\sigma_mh \\ & + d_0^3\chi_mh\sigma_i^2 - 36u_1\sigma_mE_0^4c_0^4\chi_i^2 - 30u_1\sigma_mE_0^2c_0^2\chi_i\sigma_i \\ & + 6d_0^3\chi_mhE_0^4c_0^4\chi_i^2 + 5d_0^3\chi_mhE_0^2c_0^2\chi_i\sigma_i - 6u_1\sigma_m\sigma_i^2 + 5d_0^3\chi_m\sigma_m^2h \\ & + 15\sigma_i\chi_m^2E_0^2d_0^5h) / [6\sigma_m(\sigma_m + 2E_0^2c_0^2\chi_i + 2\chi_mE_0^2d_0^2 + \sigma_i)(\sigma_i + \sigma_m \\ & + 3\chi_mE_0^2d_0^2 + 3E_0^2c_0^2\chi_i)] \}, \end{aligned}$$

where u_1 is arbitrary. Following the above analysis we choose $u_1 = -E_0$ to meet the asymptotic condition (29).

Similarly, when meeting conditions (50) and (51) via the first, second and third components of solution series, linear algebraic equations for $b_2, c_2, d_2, u_2, v_2, e_2$ are obtained.

If we denote the algebraic equations for $b_i, c_i, d_i, u_i, \dots$ as PS_i ($i \geq 2$). PS_i has two properties: first, PS_i is a group of linear algebraic equations for b_i, c_i, \dots ; second, the number of equations in PS_i is equal to the number of variables $b_i, c_i, d_i, u_i, \dots$. Hence, $b_i, c_i, d_i, u_i, \dots$ can be uniquely determined. For the limitation of this paper, we just list the solution set for $b_2, c_2, d_2, u_2, v_2, e_2$ as follows:

$$\begin{aligned}
 BC_2 = \left\{ \begin{aligned}
 b_2 &= (180E_0^2\chi_m\sigma_i b_1^2\sigma_m^2 + 180\chi_m^2\sigma_i d_0^4 h^2\sigma_m + 1620E_0^2\chi_m\sigma_i d_1^2\sigma_m^2 \\
 &\quad - 720E_0^2\chi_m^2\sigma_i b_1\sigma_m h d_0^3 - 180E_0^2\chi_i\sigma_m^2 c_1^2\sigma_i \\
 &\quad + 540E_0^2\chi_i^2\sigma_m^2 c_0^3 h e_1 - 720E_0^2\chi_m^2\sigma_i h d_0^3 d_1\sigma_m \\
 &\quad + 180E_0^2\chi_m\sigma_i u_1^2\sigma_m^2 + 485\chi_m^3\sigma_i E_0^2 h^2 d_0^6 \\
 &\quad - 1620E_0^2\chi_i\sigma_m^2 e_1^2\sigma_i + 840E_0^2\chi_m^2\sigma_i h d_0^3 u_1\sigma_m \\
 &\quad + 180\chi_i c_0 E_0^2\sigma_i h \chi_m d_0^2\sigma_m b_1 + 180\chi_m^2\sigma_i d_0^4 h\sigma_m \\
 &\quad - 360\chi_i c_0 E_0^2\sigma_i h \chi_m d_0^2 u_1\sigma_m - 60\chi_i c_0\sigma_i h^2\chi_m d_0^3\sigma_m \\
 &\quad + 180\chi_i c_0 E_0^2\sigma_i h \chi_m d_0^2 d_1\sigma_m - 84\chi_i c_0 E_0^2\sigma_i h^2\chi_m^2 d_0^5 \\
 &\quad - 60\chi_i c_0\sigma_i h \chi_m d_0^3\sigma_m) / (360\sigma_i E_0^2\sigma_m^2(\chi_m d_0 - \chi_i c_0)), \\
 c_2 &= (120\chi_m^2\sigma_i d_0^4 h^2\sigma_m + 180E_0^2\chi_m\sigma_i b_1^2\sigma_m^2 - 1620E_0^2\chi_i\sigma_m^2 e_1^2\sigma_i \\
 &\quad + 480E_0^2\chi_m^2\sigma_i h d_0^3 u_1\sigma_m + 180E_0^2\chi_m\sigma_i u_1^2\sigma_m^2 \\
 &\quad + 401\chi_m^3\sigma_i E_0^2 h^2 d_0^6 - 540E_0^2\chi_m^2\sigma_i h d_0^3 d_1\sigma_m \\
 &\quad + 1080E_0^2\chi_i^2\sigma_m^2 c_0^3 h e_1 - 540E_0^2\chi_m^2\sigma_i b_1\sigma_m h d_0^3 \\
 &\quad + 1620E_0^2\chi_m\sigma_i d_1^2\sigma_m^2 - 180E_0^2\chi_i\sigma_m^2 c_1^2\sigma_i + 120\chi_m^2\sigma_i d_0^4 h\sigma_m \\
 &\quad - 540\chi_m d_0 E_0^2 h \chi_i c_0^2 e_1\sigma_m^2) / (360\sigma_i E_0^2\sigma_m^2(\chi_m d_0 - \chi_i c_0)), \\
 e_2 &= (71\chi_m^3\sigma_i E_0^2 h^2 d_0^6 + 180E_0^2\chi_m\sigma_i b_1\sigma_m^2 d_1 - 252E_0^2\chi_m^2\sigma_i h d_0^3 d_1\sigma_m \\
 &\quad + 90E_0^2\chi_m^2\sigma_i h d_0^3 u_1\sigma_m - 60E_0^2\chi_m\sigma_i b_1\sigma_m^2 u_1 \\
 &\quad + 90E_0^2\chi_i^2\sigma_m^2 c_0^3 h e_1 - 180E_0^2\chi_i\sigma_m^2\sigma_i c_1 e_1 \\
 &\quad + 20\chi_m^2\sigma_i d_0^4 h\sigma_m + 20\chi_m^2\sigma_i d_0^4 h^2\sigma_m \\
 &\quad - 80E_0^2\chi_m^2\sigma_i b_1\sigma_m h d_0^3) / (180\sigma_i E_0^2\sigma_m^2(\chi_m d_0 - \chi_i c_0)), \\
 d_2 &= (-80E_0^2\chi_m^2\sigma_i b_1\sigma_m h d_0^3 + 108\chi_i c_0 E_0^2\sigma_i h \chi_m d_0^2 d_1\sigma_m \\
 &\quad - 5\chi_i c_0 E_0^2\sigma_i h^2\chi_m^2 d_0^5 + 90E_0^2\chi_m^2\sigma_i h d_0^3 u_1\sigma_m \\
 &\quad - 360E_0^2\chi_m^2\sigma_i h d_0^3 d_1\sigma_m + 76\chi_m^3\sigma_i E_0^2 h^2 d_0^6 \\
 &\quad + 180E_0^2\chi_m\sigma_i b_1\sigma_m^2 d_1 + 20\chi_m^2\sigma_i d_0^4 h\sigma_m \\
 &\quad - 60E_0^2\chi_m\sigma_i b_1\sigma_m^2 u_1 + 90E_0^2\chi_i^2\sigma_m^2 c_0^3 h e_1 \\
 &\quad - 180E_0^2\chi_i\sigma_m^2\sigma_i c_1 e_1 + 20\chi_m^2\sigma_i d_0^4 h^2\sigma_m) / (180\sigma_i E_0^2\sigma_m^2(\chi_m d_0 - \chi_i c_0)), \\
 u_2 &= \frac{\chi_m(-5h^2\chi_m^2 d_0^6 + 180u_1\sigma_m^2 d_1 - 132\chi_m d_0^3 h d_1\sigma_m)}{300\sigma_m^2(\chi_m d_0 - \chi_i c_0)}, \\
 v_2 &= \frac{\chi_m(-5h^2\chi_m^2 d_0^6 + 180u_1\sigma_m^2 d_1 - 132\chi_m d_0^3 h d_1\sigma_m)}{300\sigma_m^2(\chi_m d_0 - \chi_i c_0)} \left. \right\}.
 \end{aligned}
 \end{aligned}$$

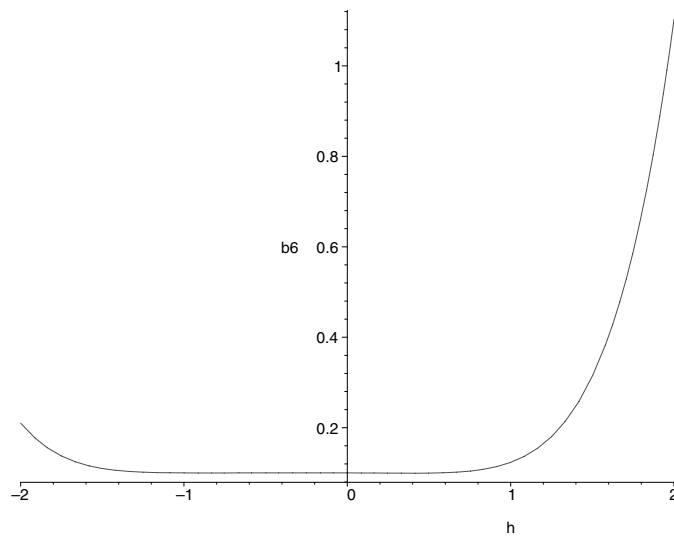


Figure 1. The h -curve of b_6 for fixed values $\sigma_i = 0.2$, $\sigma_m = 0.3$, $\chi_i = 4$ and $\chi_m = 1$.

It can be seen from the above results that only under the condition

$$(\chi_m - \chi_i)(\sigma_m - \sigma_i) < 0, \tag{67}$$

are the values of parameters $b_0, d_0, b_1, c_1, d_1, e_1, \dots$ real.

In [7], the authors consider a model in which the inclusion is a nonlinear dielectric material and the host is linear (that is $\chi_m = 0$). In this case, the nonlinear differential equations for potential ϕ can be solved exactly. When $\chi_m = 0$, our solutions can be generated as

$$\phi^m(r, \theta) = \left[-b_1 E_0^3 r + \left(b_1 E_0^3 - \sqrt{\frac{\sigma_m - \sigma_i}{\chi_i}} \right) \frac{1}{r} \right] \cos(\theta) \tag{68}$$

and

$$\phi^i(r, \theta) = -\sqrt{\frac{\sigma_m - \sigma_i}{\chi_i}} r \cos(\theta), \tag{69}$$

letting $b_1 = 1/E_0^2$, $E_0 - \sqrt{\frac{\sigma_m - \sigma_i}{\chi_i}} = B$, $-\sqrt{\frac{\sigma_m - \sigma_i}{\chi_i}} = C$. The solutions (68) and (69) are none other than the exact solutions (24) and (25) in [7].

4. Result analysis

Liao [20] proved that, as long as a solution series given by the HAM converges, it must be one of the solutions. So, it is important and necessary to ensure that our solution series are convergent. For simplicity we set $E_0 = 1$ in the following analysis. Note that the solution series (46) and (47) contain an auxiliary parameter h . Obviously, the convergence of them is determined by this auxiliary parameter. However, there are four unknown parameters $\sigma_m, \sigma_i, \chi_m$ and χ_i in our solution series. For each group given values of these parameters, h can be determined by drawing the so-called h curve; for example, when $\sigma_i = 0.2$, $\sigma_m = 0.3$, $\chi_i = 4$, $\chi_m = 1$, the h curve of b_6 is displayed in figure 1. It shows that b_6 is convergent if $-1.2 \leq h \leq 0.2$, taking the middle value $h = -0.5$ to get the fastest convergent rate.

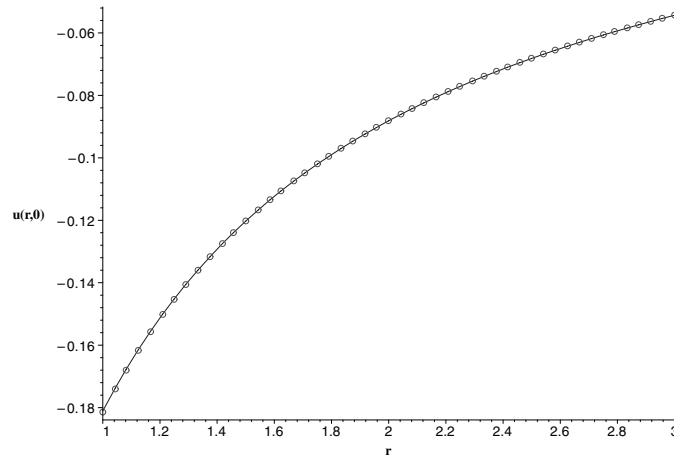


Figure 2. The analytic approximations of $\phi^m(r, 0)$ given by (46) for the case $\sigma_i = 0.2, \sigma_m = 0.3, \chi_i = 4$ and $\chi_m = 1$. Symbols: third-order approximation; solid line: sixth-order approximation.

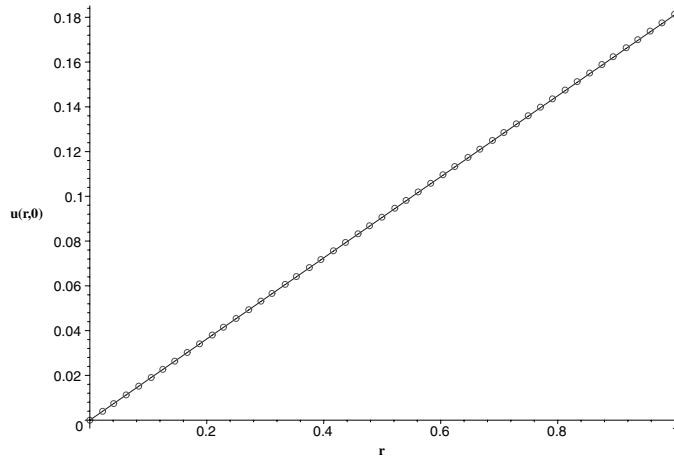


Figure 3. The analytic approximations of $\phi^i(r, 0)$ given by (47) for the case $\sigma_i = 0.2, \sigma_m = 0.3, \chi_i = 4$ and $\chi_m = 1$. Symbols: third-order approximation; solid line: sixth-order approximation.

For the parameter values $\sigma_i = 0.2, \sigma_m = 0.3, \chi_i = 4, \chi_m = 1$, we compare the third-order and the sixth-order approximations of the solution $\phi^m(r, \theta)$ in figure 2. It can be seen that the third-order approximation agrees very well with the sixth-order approximation.

Similarly, figure 3 compares the third-order approximation with the sixth-order approximation of $\phi^i(r, \theta)$. It can be seen that they agree very well.

To further verify the correctness of the solutions obtained in this paper, we substitute our analytic approximating solution $\phi^m(r, \theta)$ expressed by (48) into equation (1) to evaluate the corresponding residual error in the host region, for the same parameter values $\sigma_i = 0.2, \sigma_m = 0.3, \chi_i = 4$ and $\chi_m = 1$. The residual error of the sixth-order approximation solution $\phi^m(r, \theta)$ under $h = -0.5$ is shown in figure 4(a). Note that the maximum magnitude of the residual error is always gained when $r = 1$ (namely, at boundary $\rho = 1$). Therefore,

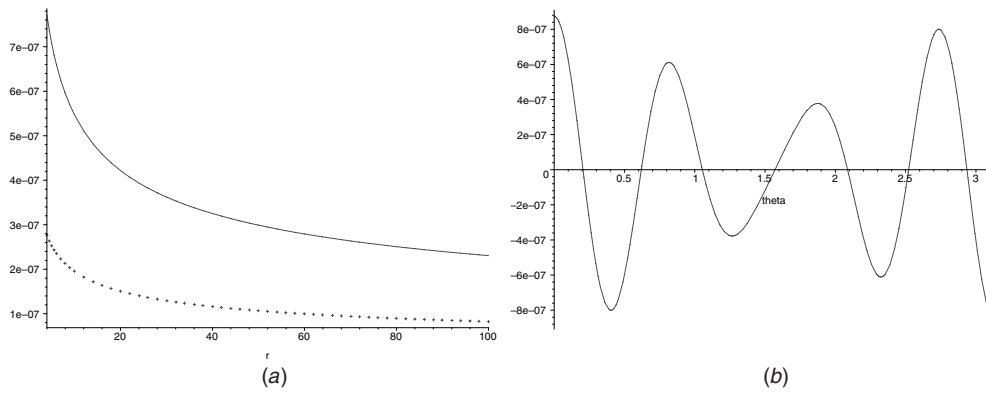


Figure 4. (a) Solid line: the residual error of the sixth-order approximation $\phi^m(r, 0)$; symbols: the residual error of the sixth-order approximation $\phi^m(r, 2)$. (b) The maximum magnitude of the residual error of the sixth-order approximation $\phi^m(r, \theta)$ at $r = 1$.

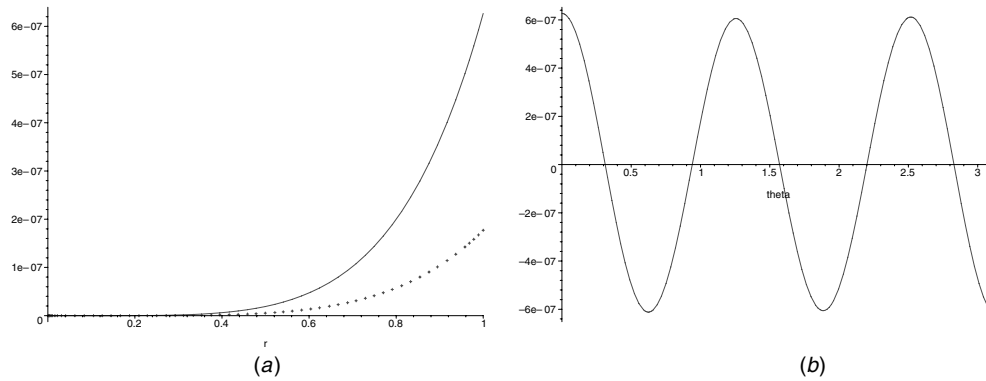


Figure 5. (a) Solid line: the residual error of the sixth-order approximation $\phi^i(r, 0)$; symbols: the residual error of the sixth-order approximation $\phi^i(r, 2)$. (b) The maximum magnitude of the residual error of the sixth-order approximation $\phi^i(r, \theta)$ at $r = 1$.

figure 4(b) shows the maximum magnitude of the residual error at $r = 1$. It can be seen that the maximum magnitude of the residual error of the sixth-order approximation $\phi^m(r, \theta)$ when $h = -0.5$ is less than 9×10^{-7} .

Similarly, we substitute the analytic approximating solution $\phi^i(r, \theta)$ expressed by (49) into equation (1) to evaluate the corresponding residual error in the inclusion region. The residual error of the sixth-order approximation $\phi^i(r, \theta)$ under $h = -0.5$ is shown in figure 5(a). The maximum magnitude of the residual error at $r = 1$ is shown in figure 5(b). It can be seen that the maximum magnitude of the residual error of the sixth-order approximation $\phi^i(r, \theta)$ when $h = -0.5$ is less than 7×10^{-7} .

For convenience when comparing our solutions with those in [21], we compute the value

$$\int_{r=0}^1 \int_{\theta=0}^{2\pi} |E^i|^2 d\theta dr, \tag{70}$$

which is a quantity with great physical significance.

From now on, for simplicity, we denote (70) as H_j when ϕ^i is replaced by the j th-order approximation of it. Several groups of the values of parameters χ_i, χ_m, σ_i and σ_m are

Table 1. The values of H_4 , H_6 and $|H_4 - H_6|$ for several group values of parameters χ_i , χ_m , σ_i , σ_m .

χ_i	σ_i	χ_m	σ_m	h	H_4	H_6	$ H_4 - H_6 $	Equation (1) type
4	0.2	1	0.3	-0.4	0.204 588 7779	0.204 588 1526	6.3×10^{-7}	Strong-strong
4	0.2	1	7	-0.02	14.108 029 56	14.10802374	5.8×10^{-6}	Weak-strong
4	0.2	0.1	0.3	-0.67	0.161 072 1248	0.161 072 0032	1.2×10^{-7}	Strong-weak
0.4	2	0.1	3	-0.7	20.436 523 60	20.436 5844	6.08×10^{-4}	Weak-weak
-4	0.2	1	-0.3	-0.1	0.617 827 2039	0.617 827 6509	4.5×10^{-7}	Strong-strong
4	-0.2	0.1	3	0.04	5.156 173 620	0.617 827 6509	3.0×10^{-8}	Weak-strong
4	0.2	-0.1	7	-0.02	10.420 342 70	10.420 345 39	2.7×10^{-5}	Strong-weak
0.4	2	-1	7	0.02	22.936 011 96	22.936 017 96	6.0×10^{-4}	Weak-weak

Table 2. The values of H_4 , H_6 and $|H_4 - H_6|$ obtained in [21] for several group values of parameters χ_i , χ_m , σ_i , σ_m .

χ_i	σ_i	χ_m	σ_m	h	H_4	H_6	$ H_4 - H_6 $	Equation (1) type
4	0.2	1	0.3	-0.09	3.570 725 796	3.570 728 909	7.1×10^{-5}	Strong-strong
4	0.2	1	7	-0.75	8.502 676 962	8.502 676 962	0	Weak-strong
4	0.2	0.1	0.3	-0.22	1.659 787 390	1.659 787 390	1.47×10^{-3}	Strong-weak
0.4	2	0.1	3	-0.926	7.701 672 968	7.701 672 968	1.2×10^{-8}	Weak-weak
-4	0.2	1	-0.3	-0.00128	562.294 1434	1259.998 267	697.7041236	Strong-strong
4	-0.2	0.1	3	-0.92	5.688 265 394	5.688 265 394	1.0×10^{-9}	Weak-strong
4	0.2	-0.1	7	-1.03	8.034760015	8.034760034	1.9×10^{-8}	Strong-weak
0.4	2	-1	7	-0.8	12.567 392 26	12.573 605 77	6.2×10^{-4}	Weak-weak

considered to indicate different types of equations, which include strong-strong, strong-weak, weak-strong and weak-weak. It is to be stressed that among them the first component shows the type of equation (1) in the host region, and the second component shows the type of equation (1) in the inclusion region.

Table 1 lists the values of H_4 , H_6 and $|H_4 - H_6|$ for different values of parameters χ_i , χ_m , σ_i and σ_m .

As mentioned in section 1, in [21] the authors first simplify the original partial differential system for an ordinary differential system by the mode expansion method. However, for simplicity they retain only the first mode to keep the lowest approximation. For the same values of conductive parameters as shown in table 1, the values of H_4 , H_6 , $|H_6 - H_4|$ obtained in [21] are displayed in table 2. It can be seen that for some values of parameters χ_m , χ_i , σ_i , σ_m , the solutions obtained in [21] significantly deviate from the true solution.

5. The effective conductivity

In this section, we investigate the effective conductivity of nonlinear composite media. The effective constitutive equation of the composite medium is the most general nonlinear relation between the average electric field and the average current

$$\bar{J} = F(\bar{E}), \tag{71}$$

where \bar{J} and \bar{E} denote, respectively, the average current and the average electric field. There may be some essential differences between linear and nonlinear composite media. In a linear composite medium, there is an exact linear relation between its average electric field and its average current. However, if a composite medium contains nonlinear components, whether the nonlinear component is the inclusion or the host, and even the nonlinear component has a simple constitutive relation $J = \sigma E + \chi|E|^2 E$, the effective constitutive equation of the composite medium is the most general nonlinear relation between the average electric field and the average current

$$\bar{J} = \sigma_e \bar{E} + \chi_e |\bar{E}|^2 \bar{E} + \eta_e |\bar{E}|^4 \bar{E} + \dots \quad (72)$$

When the effective constitutive equation can be expressed by Taylor's series both in the inclusion and in the host, equation (71) can be expressed as

$$\begin{aligned} \bar{J} = \frac{1}{V} \int_{\Omega_i} [(\sigma_i - \sigma_m)E + (\chi_i - \chi_m)|E|^2 E + (\eta_i - \eta_m)|E|^4 E + \dots] dV \\ + \sigma_m E + \chi_m |E|^2 E + \eta_m |E|^4 E + \dots \end{aligned} \quad (73)$$

From (72) and (73), we obtain

$$\begin{aligned} \frac{1}{V} \int_{\Omega_i} [(\sigma_i - \sigma_m)E + (\chi_i - \chi_m)|E|^2 E + (\eta_i - \eta_m)|E|^4 E + \dots] dV \\ = (\sigma_e - \sigma_m)E + (\chi_e - \chi_m)|E|^2 E + (\eta_e - \eta_m)|E|^4 E + \dots \end{aligned} \quad (74)$$

In this paper, we consider the nonlinear inclusion in a nonlinear host. For simplicity, we only consider the electric field along the x -direction in the inclusion, and have

$$E_x = c_0 E_0 + (3e_1 r^2 \cos 2\theta + c_1) E_0^3 + \left(5v_2 r^4 \cos 4\theta + 3e_2 r^2 \cos 2\theta + c_2 + \frac{9e_1 r^2 c_0^2 \chi_i h}{2\sigma_i} \right) E_0^5. \quad (75)$$

Retaining only the first, third and fifth powers of E_0 , we have

$$|E|^2 E_x = c_0^3 E_0^3 + (-c_0^2 c_1 - 3c_0^2 e_1 r^2 \cos 2\theta) E_0^5. \quad (76)$$

Substituting (75) and (76) into equation (74), we have

$$\begin{aligned} \chi_e = \chi_m + \rho_i [(\sigma_i - \sigma_m) \chi_m b_3 + (\chi_i - \chi_m) c_0^3], \\ \eta_e = \eta_m + \rho_i [(\sigma_i - \sigma_m) c_2 + (\chi_i - \chi_m) c_0^2 c_1], \end{aligned} \quad (77)$$

where ρ_i is the condense of the inclusion. Substituting the values of $b_3, c_j (j = 0, 1, 3)$, we can obtain the approximating expression for the effective conductivity.

6. Summary

In this paper, the homotopy analysis method is employed to compute analytically approximate solutions of nonlinear conducting composite media with boundary conditions. Unlike the traditional skills to simplify the original problem by the mode expansion method, we select a linear partial differential equation as the auxiliary linear operator. In this way, the generated higher order deformation equation is an inhomogeneous linear partial differential equation. We solve it by the method of separating variables together with the method of undetermined coefficients. Although the calculation is rather complicated, the solutions obtained in this paper have very high precision. But then, there are conductive coefficient constraints $(\chi_m - \chi_i)(\sigma_m - \sigma_i) < 0$ for the solution in this paper. How to seek more general solutions requires significant work. Finally, we investigate the effective conductivity of this problem and get the approximating expression for it. This paper shows us the validity and great potential of the homotopy analysis method for nonlinear problems in science and engineering.

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